Deterministic Treatment of the Model Error in Data Assimilation
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1. Introduction
The prediction problem in geophysical fluid dynamics typically relies on two complementary elements: the model and the data. The sequence of operations that merges model and data to obtain a possibly improved estimate of the flows state is known as data assimilation. Nevertheless, the treatment of model error in data assimilation procedures is still done, in most instances, following simple assumptions such as the absence of time correlation. Fundamental problems making difficult an adequate treatment of model error in data assimilation:

- a large variety of possible error sources (incorrect parametrizations of physical processes, numerical discretizations, unresolved scales, etc.)
- the amount of available data insufficient to realistically describe the model error statistics
- lack of a general framework for model error dynamics

OBJECTIVES
1. Identifying some general laws for the evolution of the model error dynamics (with suitable application-oriented approximations)
2. Use of these dynamical laws to prescribe the model error statistics required by DA algorithms

2. Formulation
Let assume that the model and Nature are given by:

\[ \frac{d x}{dt} = f(x, \lambda) , \quad x \approx 0 \quad f \]

- the model resolves all the relevant scales -> \( \lambda = 0 \) and \( f \)
- error in the parameter \( \lambda \neq 0 \)

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\[ \delta x(t) = M_0 \delta x_0(t) + \int_0^t \! M_0 \delta x_0(t) \, dt \]
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Linear Error

\[ \delta x(t) = \lambda(t_0) \delta x_0(t) \]

- The model error acts as a deterministic process
- The important factor controlling the evolution is \( \lambda(t_0) \)
- In view of the presence of the propagator \( M_0 \), the flow instabilities are expected to influence the model error dynamics

Model error covariance

\[ P(t) = \int_0^t \! \int_0^t \! M_0^T \delta x_0(t) \, \delta x_0'(t') \, M_0 \, dt' \, dt \]

Model error correlation

\[ \rho(t, t') = P(t) \quad \rho(t, t') = P(t) \]

These equations are NOT suitable for realistic geophysical applications - Some approximations are required

SHOR T TIME APPROXIMATION - CASE I
Approx. Model error Cov.

\[ P(t) \ll \rho(t, t') = \langle f - f \rangle^2 \]


\[ \rho(t, t') \ll \rho(t, t') = \langle f - f \rangle \]

- the model does not describe the scale given by \( \lambda(t, \lambda) \)
- assume correct parameter, \( \lambda(t, \lambda) \), and set \( \lambda = 0 \)

Error in the resolved scale

\[ \delta x(t) = \lambda(t) \delta x_0(t) + \int_0^t \! \delta x(t) \, dt'(f(x, \lambda) - f(x, \lambda)) \]

Error covariance in the resolved scale

\[ P(t) = \int_0^t \! \int_0^t \! M_0^T \delta x_0(t) \, \delta x_0'(t') \, M_0 \, dt' \, dt \]

- the correlation between i.c. and model error neglected (standard hys in DA)
- the important factor controlling the evolution is the difference between the velocity fields \( f(x, \lambda) - f(x, \lambda) \)

2.2 CASE II - Error due to unresolved scales

ASSUMPTIONS:
- use of the analysis increments of a reanalysis data-set
- use of the multiplicative inflation. - Carrassi and Vannitsem (2011b)

Experiments with Lorenz (1996) model. 360 variables - two scales. - Comparison with EKF using multiplicative inflation. - Carrassi and Vannitsem (2011b)

3. Results
3.1 Sequential Data Assimilation - EKF

- EKF Forecast Error Covariance Update:

\[ P'(t) = M^T P M + \rho \]

- Model Error Covariance Matrix

Estimate \( P'(t) \) using the short time approximation

CASE I - Parametric Error

\[ \rho(t, t') \ll \rho(t, t') \ll \rho(t, t') \]

Two solutions proposed to estimate \( \rho(t, t') \):

1. Statistically based on a priori information – Short Time EKF (ST-EKF)
2. Dynamically (on the fly) using a state/parameter estimation approach – Short Time Augmented EKF (STA-EKF)


3.2 Variational Data Assimilation - 4DVAR

- analysis state as the minimum of a cost-function:

\[ \int_{t_1}^{t_0} \left( \frac{1}{2} \langle f - f \rangle + \frac{1}{2} \langle f - f \rangle \right) + \frac{1}{2} \langle f - f \rangle \]

- Model error covariances/correlations using \( P(t, t') = Q \Rightarrow \rho(t, t') \ll \rho(t, t') \)

Experiments with Lorenz (1963) model - Carrassi and Vannitsem (2010)

- Strong-constraint
- Short-time weak constraint 4DVAR
- Weak constraint 4DVAR with uncorrelated model error: \( \rho(t) = 0 \)
- Weak constraint 4DVAR with uncorrelated model error: \( \rho(t) = 0 \)

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Key References
Carrassi A. and S. Vannitsem, 2011b: Treatment of the error due to unresolved scales in sequential data assimilation. I. J. B. C., 21: 3619-3626

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