Abstract
Understanding the consequences of emissions changes from a source region into a target region is a key factor for decisions making. The source-target relationship is commonly studied by the sensitivity analysis of a response function with respect to the source. The base approach determines the sensitivity by setting out multiple-scenarios with variations of source parameter. The variability over the receptor is considered as the sensitivity. By definition, the sensitivity of a given response function with respect to the source is the gradient of that response function with respect to the source parameter. A straightforward approach over the receptor is to consider the transport and diffusion of the pollutant. Both approaches assume that the transport velocity and the initial distribution of the pollutant are known. However, they are given in the form of a data assimilation problem where imperfect initial, but not limited to, the pollutant source, meteorological model and physical measurements. As a consequence, the sensitivity analysis should be carried out on the output variable of the Data Assimilation problem. This leads to a non-standard problem on a second order adjoint system where the solution requires the third order adjoint.

Introduction
The ability to know in advance and with accuracy the effect of the change of pollutant emission from a foreign region onto a target region would help decision makers to improve public health and protect the environment. Also, the knowledge of the correct location of an emission source that could impact a target region within a given time scale is an asset in choosing the right place for some events or the storage location for delicate products. Studies show that the change in pollutant emission from a foreign region could have a significant impact on a target or response region, even at the intercontinental level. This is the case for the degradation of the air quality over remote continents [Wilhelmsen, 2004; Bisaro 2004; Holloway 2001; Almeida 2007]. Some air pollution episodes in the US were reported to be associated with transpacific transport events [Jaffe 1999; Jaffe 2003]. During the last decades, ground based measurements and satellites remote sensing have provided evidence for foreign influence of pollutants [Allan 2004; Jaffe 2003; Stohl 2004]. The time scale for the impact of a remote source on a receptor region is highly dependent on weather conditions. [Jaffe 2003] notes that under certain weather conditions, Asian emissions can be transported to North America within 5 to 8 days. As outlined by [Freng 2004; Dvorez 2004; Janow 2004; Freng 2004], attributing a given pollution episode to a specific source is complicated by the interplay of processes influencing the transport (export from the source region, the evolution in the boundary layer and the deposition losses, dilution and mixing with local sources over the receptor region). As it is difficult to make a micro-receptor relationship from observation, such analyses rely on models. [Freng 2003] uses a set of 21 numerical models to estimate the intercontinental source-receptor-relationship for smog pollution, [Wang 2004] made a similar study based on a high-resolution model.

Sensitivity analysis: First order adjoint method

- Evolution of the pollutant:
  \[ \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{\partial}{\partial x} (k \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y} (k \frac{\partial c}{\partial y}) + Q \]
  \[ c(x, y, t) \]
  \[ \Omega \]
  \[ u, v \]
  \[ k \]
  \[ Q \]
- Sensitivity problem: response function
  \[ \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{\partial}{\partial x} (k \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y} (k \frac{\partial c}{\partial y}) + Q - \frac{\partial}{\partial x} (\frac{\partial c}{\partial x}) + \frac{\partial}{\partial y} (\frac{\partial c}{\partial y}) \]
  \[ \Omega \]
  \[ u, v \]
  \[ k \]
  \[ Q \]

Example 1: one dimensional transport-diffusion problem
We consider the problem of pollution by a single species that is produced by a point source defined on the physical space \( H = [-1, 1] \). The time interval of interest is [0, T]. The pollution concentration evolves according to the one-dimensional transport-diffusion equation (5).

- Sensitivity analysis of \( c \) with respect to \( S \):
  \[ \nabla \psi (c) = \Lambda \]
  \[ \Omega \]
  \[ \Lambda \]
  \[ S \]

Numerical results

Example 2: shallow water pollution problem
We consider the problem of pollution by multiple species under the dynamics of a shallow water:

- Numerical results

Limitations

The first order adjoint approach necessitates the state of the geophysical system and the initial state of the pollutant. The state of the geophysical system and the initial state of the tracer are usually given by the solution of inverse problems using the Data Assimilation techniques. These techniques use the solutions and observations of the pollutant. The first order adjoint is then insufficient for the sensitivity analysis.

Second order adjoint for the sensitivity analysis

Model: System and pollutant evolution

- Determination of the initial conditions
  \[ \frac{\partial X}{\partial t} + U \frac{\partial X}{\partial x} + V \frac{\partial X}{\partial y} = \frac{\partial}{\partial x} (k \frac{\partial X}{\partial x}) + \frac{\partial}{\partial y} (k \frac{\partial X}{\partial y}) + Q \]
  \[ \Omega \]
  \[ U, V \]
  \[ k \]
  \[ Q \]

Evaluations of the sensitivities with respect to the sources

Response function: \( \Phi(C) = \int_{\Omega} \int_{D} c \delta(x, y, t) dt dx \)

Sensitivity of \( \Phi \) with respect to \( S \):

Solution of the non standard problem

For simplicity, let us consider the following system of two equations:

- Solution of the non standard problem

Second order adjoint for the sensitivity analysis

Model: System and pollutant evolution

- Determination of the initial conditions
  \[ \frac{\partial X}{\partial t} + U \frac{\partial X}{\partial x} + V \frac{\partial X}{\partial y} = \frac{\partial}{\partial x} (k \frac{\partial X}{\partial x}) + \frac{\partial}{\partial y} (k \frac{\partial X}{\partial y}) + Q \]
  \[ \Omega \]
  \[ U, V \]
  \[ k \]
  \[ Q \]